Grassmann–Gaussian integrals and generalized star products

2009 J. Phys. A: Math. Theor. 42 304019

(http://iopscience.iop.org/1751-8121/42/30/304019)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 304019 (16pp) doi:10.1088/1751-8113/42/30/304019

Grassmann–Gaussian integrals and generalized star products

Shahane Khachatryan[1], Robert Schrader[2] and Ara Sedrakyan[1]

[1] Yerevan Physics Institute, Br Alikhanyan 2, Yerevan 36, Armenia
[2] Institut für Theoretische Physik Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany

E-mail: shah@yerphi.am, schrader@physik.fu-berlin.de and sedrak@nbi.dk

Received 12 March 2009, in final form 2 June 2009
Published 14 July 2009
Online at stacks.iop.org/JPhysA/42/304019

Abstract
In quantum scattering on networks, there is a nonlinear composition rule for on-shell scattering matrices which serves as a replacement for the multiplicative rule of transfer matrices valid in other physical contexts. In this paper, we show how this composition rule is obtained using Berezin integration theory with Grassmann variables.

PACS number: 73.21.Hb
Mathematics Subject Classification: 81U20, 05C50, 34L25

In memory of Al B Zamolodichikov

1. Introduction

Potential scattering for one particle Schrödinger operators on the line possesses a remarkable property concerning its (on-shell) scattering matrix given as a 2×2 matrix-valued function of the energy. Let the potential V be given as the sum of two potentials V_1 and V_2 with disjoint support. Then the scattering matrix for V at a given energy is obtained from the two scattering matrices for V_1 and V_2 at the same energy by a certain nonlinear, noncommutative but associative composition rule. This fact in quantum scattering theory on the line has been discovered independently by several authors (see, e.g. [1, 21, 24, 27, 28] and is an easy consequence of the multiplicative property of the transfer matrix of the Schrödinger equation (see, e.g. [17]). It has also been well known in the theory of mesoscopic systems and multichannel conductors (see, e.g. [2, 6–10, 12, 23, 31, 32]). In higher space dimensions a similar rule is unlikely to exist due to the defocusing of wave packets under propagation. However, for large separation between the supports of the potentials the scattering matrix at a given energy may asymptotically be expressed in terms of the scattering matrices for V_1 and V_2 at the same energy [15–17].

To the best of our knowledge the composition rule for 2×2 scattering matrices was first observed in the context of electric network theory by Redheffer [25, 26], who called it the *star product*. Now there are situations, where the concept of a transfer matrix cannot always be introduced but where the (on-shell) scattering matrix nevertheless exists. An important example is given by quantum-dynamical models on graphs, that is quasi-one-dimensional quantum systems and which are described by Schrödinger operators. Such systems are nowadays a subject of intensive study (see, e.g. [4, 11, 22] and references quoted there). In the article [18], a composition rule, called the *generalized star product*, was introduced and further analyzed in [19]. This composition rule extends the star product of Redheffer and allows one to obtain the on-shell scattering matrix on a given graph from the on-shell scattering matrices associated with sub-graphs. The generalized star product is defined for arbitrary matrices but for unitary matrices the outcome is also unitary.

In this paper, we provide a new way of obtaining the generalized star product. The method is based on the Grassmannian (fermionic) integration theory given by Berezin [3] and it evaluates certain Gauss–Grassmann integrals. In addition, we also show how one can arrive at the generalized star product using ordinary Gaussian (bosonic) distributions. Then, however, one has to work with a restriction, the covariances have to be positive-definite matrices.

This technique of using Gaussian integration with fermionic fields permits an action formulation of some network models. Thus for example it can be applied to the Chalker–Coddington network model [5] to describe plateau transitions in the quantum Hall effect [29]. The method is also very convenient for an investigation of models with a large number of scattering centers. In this limit, as well as at the critical point, one can give an equivalent quantum field theoretic formulation of network models [30].

The paper is organized as follows. In the following section in order to establish notation and for the convenience of the reader we briefly recall the basic notions of Berezin's theory. Though most of the material can be found in standard text books of quantum field theory, see, e.g. [14, 33], the relations we need seem not to be so easily accessible. In section 4, we show how to obtain the generalized star product using Gauss–Grassmann integrals. In the appendix we briefly discuss the corresponding bosonic version, that is how the generalized star product can be obtained from the standard theory of Gaussian distributions.

2. Preliminaries

In this section, we first briefly review the Grassmann integration and then we recall some concepts from graph theory that we will need.

2.1. Grassmann integration

In this subsection we briefly recall the basic notions concerning Grassmann variables and the associated integration theory, see [3], and which we will need. Let $\overline{a}_i, a_i$ be Grassmann variables, which means they anti-commute

$$\overline{a}_i\overline{a}_j = -\overline{a}_j\overline{a}_i, \qquad \overline{a}_i a_j = -a_j\overline{a}_i, \qquad a_i a_j = -a_j a_i.$$

We denote the associated (complex) Grassmann algebra with unit $\mathbb{I}$ by $\mathcal{A}_I$. Elements α in $\mathcal{A}_I$ have a unique representation in the form

$$\alpha = \alpha(\overline{a}, a) = \sum_{J,K\subseteq I} c_{J,K}\overline{a}_J a_K, \qquad c_{J,K} \in \mathbb{C} \tag{2.1}$$

with the anti-ordered and ordered products

$$\overline{a}_J = \overset{\curvearrowleft}{\prod}_{j\in J} \overline{a}_j, \qquad a_K = \overset{\curvearrowright}{\prod}_{j\in K} a_j, \qquad J \neq \emptyset, K \neq \emptyset$$

and $\overline{a}_\emptyset = a_\emptyset = \mathbb{I}$. By definition

$$\alpha(\overline{a} = 0, a = 0) = c_{J=\emptyset, K=\emptyset}. \tag{2.2}$$

Correspondingly the subalgebra generated by the elements $\overline{a}_j, a_j$ with $i \in J$ will be denoted by $\mathcal{A}_J$. In addition we introduce symbols $\mathrm{d}\overline{a}_i$ and $\mathrm{d}a_i$ which anti-commute among themselves and with $\overline{a}_i, a_i$ and define the anti-ordered and ordered products

$$\mathrm{d}\overline{a}_J = \overset{\curvearrowleft}{\prod}_{j\in J} \mathrm{d}\overline{a}_j, \qquad \mathrm{d}a_K = \overset{\curvearrowright}{\prod}_{j\in K} \mathrm{d}a_j.$$

(Berezin-)Integration over $L \subseteq I$ is defined as

$$\int \overline{a}_J a_K \,\mathrm{d}\overline{a}_L \,\mathrm{d}a_L = \begin{cases} 0 & L \not\subseteq J \quad \text{or} \quad L \not\subseteq K \\ (-1)^{|L|} \operatorname{sign}^L_{JK} \overline{a}_{J\setminus L} a_{K\setminus L} & L \subseteq J \quad \text{and} \quad L \subseteq K \end{cases} \tag{2.3}$$

where $\operatorname{sign}^L_{JK} = \pm 1$ is defined when $L \subseteq J$ and $L \subseteq K$ and is such that

$$\overline{a}_J a_K = \operatorname{sign}^L_{JK} \overline{a}_{J\setminus L} a_{K\setminus L} \overline{a}_L a_L$$

holds. We call $\mathrm{d}\overline{a}_L \mathrm{d}a_L$ the *volume form* associated with the index set L. From definition (2.3) it is easy to see that integration may be performed in steps (Fubini's theorem), an observation which will become important in our discussion. As a special case of (2.3) and with $\alpha \in \mathcal{A}_I$ written in the form (2.1)

$$\int \alpha \,\mathrm{d}\overline{a}_I \,\mathrm{d}a_I = (-1)^{|I|} c_{I,I}. \tag{2.4}$$

In analogy to the Lebesgue integral over $\mathbb{R}^n$ this integral exhibits a *translation invariance* in the following form. Introduce additional Grassmann variables $\overline{b}_i, b_i$, again with the index i in I and which anti-commute with all the previous Grassmann variables. With $\alpha(\overline{a}, a)$ as in (2.1), by $\alpha(\overline{a} - \overline{b}, a - b)$ we understand the expression

$$\alpha(\overline{a} - \overline{b}, a - b) = \sum_{J,K\subseteq I} c_{J,K} (\overline{a} - \overline{b})_J (a - b)_K \tag{2.5}$$

with the anti-ordered and ordered products

$$(\overline{a} - \overline{b})_J = \overset{\curvearrowleft}{\prod}_{j\in J} (\overline{a}_j - \overline{b}_j), \qquad (a - b)_K = \overset{\curvearrowright}{\prod}_{j\in K} (a - b)_j.$$

Expanding each $(\overline{a} - \overline{b})_J (a - b)_K$ into a sum of monomials $\overline{a}_{J'} a_{K'}$ it is easy to establish translation invariance of the integral in the form

$$\int \alpha(\overline{a} - \overline{b}, a - b) \,\mathrm{d}\overline{a}_I \,\mathrm{d}a_I = \int \alpha(\overline{a}, a) \,\mathrm{d}\overline{a}_I \,\mathrm{d}a_I$$

or even more generally

$$\int \alpha(\overline{a} - \overline{b}, a - b; \overline{b}, b, \overline{c}, c) \,\mathrm{d}\overline{a}_I \,\mathrm{d}a_I = \int \alpha(\overline{a}, a; \overline{b}, b, \overline{c}, c) \,\mathrm{d}\overline{a}_I \,\mathrm{d}a_I, \tag{2.6}$$

valid as a relation in $\mathcal{B}$, the Grassmann algebra generated by $\overline{b}_i$ and b_i and possibly additional Grassmann variables $\overline{c}_k$ and c_k. This seemingly trivial relation will also become very useful below. Let A be any complex $n \times n$ matrix and choose $I_n = \{1, 2, \ldots, n\}$ to be the index set. Set

$$\bar{a} \cdot Aa = \sum_{1 \leqslant i,j \leqslant n} \bar{a}_i A_{ij} a_j .$$

The Gauss–Grassmann integral can be calculated as

$$\int \mathrm{e}^{-\bar{a}\cdot Aa} \, \mathrm{d}\bar{a}_{I_n} \, \mathrm{d}a_{I_n} = \det A, \tag{2.7}$$

the Gaussian distribution analogue of which is relation (A.2) in the appendix. In a further analogy to ordinary Gaussian distributions, see (A.3) below, we obtain

$$\begin{aligned} \int \bar{a}_i \, \mathrm{e}^{-\bar{a}\cdot Aa} \, \mathrm{d}\bar{a}_{I_n} \, \mathrm{d}a_{I_n} &= \int a_i \, \mathrm{e}^{-\bar{a}\cdot Aa} \, \mathrm{d}\bar{a}_{I_n} \, \mathrm{d}a_{I_n} = 0 \\ \int a_i \bar{a}_j \, \mathrm{e}^{-\bar{a}\cdot Aa} \, \mathrm{d}\bar{a}_{I_n} \, \mathrm{d}a_{I_n} &= \det A \, A^{-1}_{ij}. \end{aligned} \tag{2.8}$$

We need an extension of (2.7) when $\det A \neq 0$. Set

$$\bar{b} \cdot a = -a \cdot \bar{b} = \sum_{1 \leqslant i \leqslant n} \bar{b}_i a_i , \qquad \bar{a} \cdot b = -b \cdot \bar{a} = \sum_{1 \leqslant i \leqslant n} \bar{a}_i b_i .$$

Then one can prove

$$\int \mathrm{e}^{-\bar{a}\cdot Aa + \bar{b}\cdot a + \bar{a}\cdot b} \, \mathrm{d}\bar{a}_{I_n} \, \mathrm{d}a_{I_n} = \det A \, \mathrm{e}^{\bar{b}\cdot A^{-1} b} \tag{2.9}$$

by using the translation invariance (2.6) of the integral and by completing the square in the exponent. Finally we provide a variant of (2.9), which will become important below. For any $1 \leqslant p \leqslant n$ consider I_p as a subset of I_n and let $I_p^c = \{p+1, \ldots, n\}$ be its complement. Let $\bar{a}^{(1)}, a^{(1)}$ denote the set of Grassmann variables $\bar{a}_i, a_1$ with $1 \leqslant i \leqslant p$ and $\bar{a}^{(2)}, a^{(2)}$ those with $p+1 \leqslant i \leqslant n$. For a matrix A as before, consider the corresponding matrix block decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{2.10}$$

where A_{11} is a $p \times p$ matrix, A_{12} a $p \times (n-p)$ matrix, etc and set

$$\begin{aligned} \bar{a}^{(1)} \cdot A_{11} a^{(1)} &= \sum_{1 \leqslant i,j \leqslant p} \bar{a}_i A_{ij} a_j , & \bar{a}^{(1)} \cdot A_{12} a^{(2)} &= \sum_{1 \leqslant i \leqslant p,\, p+1 \leqslant j \leqslant n} \bar{a}_i A_{ij} a_j \\ \bar{a}^{(2)} \cdot A_{21} a^{(1)} &= \sum_{p+1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant p} \bar{a}_i A_{ij} a_j , & \bar{a}^{(2)} \cdot A_{22} a^{(2)} &= \sum_{p+1 \leqslant i,j \leqslant n} \bar{a}_i A_{ij} a_j \end{aligned} \tag{2.11}$$

such that

$$\bar{a} \cdot Aa = \bar{a}^{(1)} \cdot A_{11} a^{(1)} + \bar{a}^{(1)} \cdot A_{12} a^{(2)} + \bar{a}^{(2)} \cdot A_{21} a^{(1)} + \bar{a}^{(2)} \cdot A_{22} a^{(2)} \tag{2.12}$$

holds.

Lemma 2.1. *Let the $(n-p) \times (n-p)$ matrix A_{22} be invertible and set*

$$\widehat{A} = A_{11} - A_{12} A_{22}^{-1} A_{21},$$

a $p \times p$ matrix. Then

$$\int \mathrm{e}^{-\bar{a}\cdot Aa} \, \mathrm{d}\bar{a}_{I_n^c} \, \mathrm{d}a_{I_n^c} = \det A_{22} \cdot \mathrm{e}^{-\bar{a}^{(1)}\cdot \widehat{A} a^{(1)}} \tag{2.13}$$

holds. Therefore the relation

$$\det A = \det A_{22} \cdot \det \widehat{A} \tag{2.14}$$

is valid. If in addition $\widehat{A}$ *is invertible, the matrix elements of its inverse are the corresponding ones for* A^{-1}

$$\widehat{A}^{-1}_{ij} = A^{-1}_{ij}, \qquad 1 \leqslant i, j \leqslant p. \tag{2.15}$$

Proof. We use the decomposition (2.12) and complete the square to obtain

$$\mathrm{e}^{-\overline{a}\cdot Aa} = \mathrm{e}^{-(\overline{a}^{(2)}+A_{22}^{-1T}A_{21}^{T}\overline{a}^{(1)})\cdot A_{22}(a^{(2)}+A_{22}^{-1}A_{21}a^{(1)})}\,\mathrm{e}^{-\overline{a}^{(1)}\cdot\widehat{A}a^{(1)}}$$

and (2.13) follows from (2.7) and translation invariance. Equation (2.14) in turn follows from (2.7) by integrating (2.13) out over the Grassmann variables $\overline{a}_i, a_i$ with $i \in I_p$. As for the last claim, if $\widehat{A}$ is invertible, then so is A by (2.14) (and conversely under the present assumption that A_{22} is invertible). By the last relation in (2.8) and by (2.13) for all $1 \leqslant i, j \leqslant p$

$$\begin{aligned}\det\widehat{A}\cdot\widehat{A}^{-1}_{ij} &= \int a_i\overline{a}_j\,\mathrm{e}^{-\overline{a}\cdot\widehat{A}a}\,\mathrm{d}\overline{a}_{I_p}\,\mathrm{d}a_{I_p}\\ &= \frac{1}{\det A_{22}}\int a_i\overline{a}_j\,\mathrm{e}^{-\overline{a}\cdot Aa}\,\mathrm{d}\overline{a}_{I_n}\,\mathrm{d}a_{I_n} = \frac{\det A}{\det A_{22}}\cdot {A^{-1}}_{ij}\end{aligned}$$

and (2.15) follows from (2.14). □

Remark 2.2. If A_{22} is not invertible, the left-hand side of (2.13) is still well defined but not the right-hand side. Relation (2.14) also follows from the factorization of a block matrix. It involves the *Schur complement* of A_{22}, which is just $\widehat{A}$, see, e.g. [13, 34]. In addition the inverse of the Schur complement enters the inverse of A as one block part and this is just relation (2.15)

$$\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}^{-1} = \begin{pmatrix} \widehat{A}^{-1} & -\widehat{A}^{-1}A_{12}A_{22}^{-1}\\ -A_{22}^{-1}A_{21}\widehat{A}^{-1} & A_{22}^{-1}+A_{22}^{-1}A_{21}\widehat{A}^{-1}A_{12}A_{22}^{-1}\end{pmatrix}. \tag{2.16}$$

2.2. Some basic concepts from graph theory

We first recall some notions, which will be useful in the following. A finite noncompact graph is a 4-tuple $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial)$, where $\mathcal{V}$ is a finite set of *vertices*, $\mathcal{I}$ is a finite set of *internal edges* and $\mathcal{E}$ is a finite set of *external edges*. Set

$$n(\mathcal{V}) = |\mathcal{V}|, \qquad n(\mathcal{I}) = |\mathcal{I}|, \qquad n(\mathcal{E}) = |\mathcal{E}|,$$

the number of elements in these sets. We assume each of these sets $\mathcal{V}, \mathcal{E}, \mathcal{I}$ to be ordered in some arbitrary but fixed way. This induces an ordering in $\mathcal{E}\cup\mathcal{I}$, where by definition elements in $\mathcal{E}$ come first. On the product set $(\mathcal{E}\cup\mathcal{I})\times\mathcal{V}$ by definition the induced ordering $\preceq$ is such $(i, v) \preceq (i', v')$ if and only if $v \prec v'$ or $v = v'$ and $i \preceq i'$. Elements in $\mathcal{I}\cup\mathcal{E}$ are called *edges*. The map ∂ assigns to each internal edge $i \in \mathcal{I}$ an ordered pair of (possibly equal) vertices $\partial(i) := \{v_1, v_2\}$ and to each external edge $e \in \mathcal{E}$ a single vertex v. The vertices $v_1 =: \partial^-(i)$ and $v_2 =: \partial^+(i)$ are called the *initial* and *final* vertices of the internal edge i, respectively. The vertex $v = \partial(e)$ is the initial vertex of the external edge e. If $\partial(i) = \{v, v\}$, that is, $\partial^-(i) = \partial^+(i)$ then i is called a *tadpole*. However, to facilitate our exposition, we will exclude tadpoles. Two vertices v and v' are *adjacent*, if there is $i \in \mathcal{I}$ such that these vertices form $\partial(v)$. To any $v \in \mathcal{V}$ we associate the set of edges terminating at v, $\mathcal{I}(v) = \{i \in \mathcal{E}\cup\mathcal{I} | v \in \partial(i)\}$.

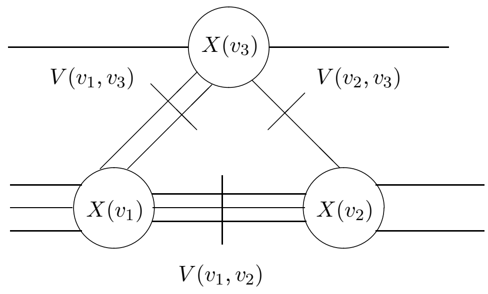

Figure 1. A graph with three vertices, seven external and six internal edges. $X(v_1)$ is a 8×8, $X(v_2)$ a 6×6 and $X(v_3)$ a 5×5 matrix. $V(v_1, v_2)$ is a invertible 3×3 matrix, $V(v_2, v_3)$ is just a non-vanishing complex number, while $V(v_1, v_3)$ is an invertible 2×2 matrix.

We set $n(v) = |\mathcal{I}(v)|$, the number of edges terminating at v. Each of these sets inherits an ordering from the ordering of $\mathcal{E} \cup \mathcal{I}$. Also

$$n(v, v') = n(v', v) = |\mathcal{I}(v) \cap \mathcal{I}(v')|, \qquad v \neq v'$$

is the number of (internal) edges connecting v with v', so $n(v, v') \leqslant \min(n(v), n(v'))$. $n(v, v')$ is called the *connectivity matrix* of the graph $\mathcal{G}$. $\mathcal{I}(v) \cap \mathcal{I}(v') \subseteq \mathcal{I}$ holds for $v \neq v'$ and likewise this set inherits an ordering from the ordering of $\mathcal{I}$. As (unordered) sets each $\mathcal{I}(v)$ is a disjoint union

$$\mathcal{I}(v) = (\mathcal{I}(v) \cap \mathcal{E}) \cup_{v':v' \neq v} (\mathcal{I}(v) \cap \mathcal{I}(v')).$$

By definition, a graph is *compact* if $\mathcal{E} = \emptyset$, otherwise it is *noncompact*. Throughout the whole work we will assume that the graph $\mathcal{G}$ is connected, that is, for any $v, v' \in V$ there is an ordered sequence $\{v_1 = v, v_2, \ldots, v_n = v'\}$ such that any two successive vertices in this sequence are adjacent. In particular, this implies that any vertex of the graph $\mathcal{G}$ has nonzero degree, i.e., for any vertex v there is at least one edge with which it is incident, $n(v) > 0$. In addition $n(\mathcal{I}) \geqslant n(\mathcal{V}) - 1$ is valid. By definition a *star graph* is a connected graph which has no internal edges, only one vertex and at least one external edge.

A graph can be equipped as follows with a metric structure. To any internal edge $i \in \mathcal{I}$ we associate an interval $[0, a_i]$ with $a_i > 0$ such that the initial vertex of i corresponds to $x = 0$ and the terminal one—to $x = a_i$. Any external edge $e \in \mathcal{E}$ will be associated with a semi-line $[0, \infty)$ such that $\partial(e)$ corresponds to $x = 0$. We call the number a_i the *length* of the internal edge i. The set of lengths $\{a_i\}_{i \in \mathcal{I}}$, which will also be treated as an element of $\mathbb{R}^{|\mathcal{I}|}$, will be denoted by $\underline{a}$. A compact or noncompact graph $\mathcal{G}$ endowed with a metric structure is called a *metric graph* $(\mathcal{G}, \underline{a})$.

3. The fermionic construction

Given a graph $\mathcal{G}$, we introduce the following data. To each vertex v we associate a complex $n(v) \times n(v)$ matrix $X(v)$ indexed by the set $\mathcal{I}(v)$. We call $X(v)$ a *vertex matrix*. In addition, complex, invertible $n(v, v') \times n(v, v')$ matrices $V(v, v')$ are given for any pair $v \neq v'$. The associated index sets are $\mathcal{I}(v) \cap \mathcal{I}(v')$ and they are supposed to satisfy

$$V(v, v') = V(v', v)^{-1}. \tag{3.1}$$

$V(v, v')$ is called a *connecting matrix* between v and v'. We write $\underline{X}_{\mathcal{G}} = \{X(v)\}_{v \in \mathcal{V}}$ and $\underline{V}_{\mathcal{G}} = \{V(v, v')\}_{v \neq v' \in \mathcal{V}}$ for these two sets of data. Figure 1 gives a pictorial description for a graph with three vertices.

Remark 3.1. Within the context of scattering on quantum graphs, see [18, 19] $X(v)$ is the unitary scattering matrix $S(v; E)$ at energy $E > 0$ on a single-vertex graph with $n(v)$ external edges and $V(v, v')$ is a unitary diagonal matrix

$$V(v, v') = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\sqrt{E}a_i})_{i\in\mathcal{I}(v)\cap\mathcal{I}(v')}. \tag{3.2}$$

We also introduce a total of $2(|\mathcal{E}| + 2|\mathcal{I}|)$ anti-commuting Grassmann variables

$$\begin{aligned} &\overline{\eta}_{e,v}, \eta_{e,v}, \mathrm{d}\overline{\eta}_{e,v}, \mathrm{d}\eta_{e,v}, && e \in \mathcal{E}, v = \partial(e) \\ &\overline{\eta}_{i,v}, \eta_{i,v}, \mathrm{d}\overline{\eta}_{i,v}, \mathrm{d}\eta_{i,v}, && i \in \mathcal{I}, v \in \partial(i) \end{aligned} \tag{3.3}$$

and the associated volume forms given as the anti-ordered and ordered products

$$\begin{aligned} \mathrm{d}\overline{\eta}_{\mathcal{E}}\, \mathrm{d}\eta_{\mathcal{E}} &= \overset{\curvearrowleft}{\prod}_{e\in\mathcal{E},v=\partial(e)} \mathrm{d}\overline{\eta}_{e,v} \overset{\curvearrowright}{\prod}_{e\in\mathcal{E},v=\partial(e)} \mathrm{d}\eta_{e,v} \\ \mathrm{d}\overline{\eta}_{\mathcal{I}}\, \mathrm{d}\eta_{\mathcal{I}} &= \overset{\curvearrowleft}{\prod}_{i\in\mathcal{I},v\in\partial(i)} \mathrm{d}\overline{\eta}_{i,v} \overset{\curvearrowright}{\prod}_{i\in\mathcal{I},v\in\partial(i)} \mathrm{d}\eta_{i,v} \\ \mathrm{d}\overline{\eta}_{\mathcal{E}\cup\mathcal{I}}\, \mathrm{d}\eta_{\mathcal{E}\cup\mathcal{I}} &= \mathrm{d}\overline{\eta}_{\mathcal{E}}\, \mathrm{d}\eta_{\mathcal{E}}\, \mathrm{d}\overline{\eta}_{\mathcal{I}}\, \mathrm{d}\eta_{\mathcal{I}} = \mathrm{d}\overline{\eta}_{\mathcal{E}}\, \mathrm{d}\overline{\eta}_{\mathcal{I}}\, \mathrm{d}\eta_{\mathcal{E}}\, \mathrm{d}\eta_{\mathcal{I}} = \mathrm{d}\overline{\eta}_{\mathcal{E}}\, \mathrm{d}\overline{\eta}_{\mathcal{I}}\, \mathrm{d}\eta_{\mathcal{I}}\, \mathrm{d}\eta_{\mathcal{E}}. \end{aligned} \tag{3.4}$$

Recall that $\mathcal{I}(v)\cap\mathcal{E}$ may be non-empty, so in order to have a compact notation we have added the index $v = \partial(e)$ in the definition of $\overline{\eta}_{e,v}$. On the other hand, $\mathcal{I}(v)\cap\mathcal{I}(v')\cap\mathcal{E}$ is always empty. For further reference we write

$$(\mathcal{E}\cup\mathcal{I}) \triangleright \mathcal{V} = \{i, v\}_{i\in\mathcal{I}(v),v\in\mathcal{V}} \subset (\mathcal{E}\cup\mathcal{I}) \times \mathcal{V} \tag{3.5}$$

for this index set and with the ordering induced by that of $(\mathcal{E}\cup\mathcal{I}) \times \mathcal{V}$. Set

$$\begin{aligned} \overline{\eta}\cdot\mathcal{L}(v)\eta &= \sum_{i,j\in\mathcal{I}(v)} \overline{\eta}_{i,v} X(v)_{ij}\eta_{j,v} \\ \overline{\eta}\cdot\mathcal{L}(v, v')\eta &= -\sum_{i,j\in\mathcal{I}(v)\cap\mathcal{I}(v')} \overline{\eta}_{i,v} V(v, v')_{ij}\eta_{j,v'} \end{aligned} \tag{3.6}$$

with the convention that $\overline{\eta}\cdot\mathcal{L}(v, v')\eta = 0$ if $\mathcal{I}(v)\cap\mathcal{I}(v') = \emptyset$. Define the *quadratic, fermionic Lagrangian* as

$$\overline{\eta}\cdot\mathcal{L}_{\mathcal{G}}\eta = \sum_v \overline{\eta}\cdot\mathcal{L}(v)\eta + \sum_{v\neq v'} \overline{\eta}\cdot\mathcal{L}(v, v')\eta. \tag{3.7}$$

We decompose the set of Grassmann variables into *exterior* and *interior variables*

$$\begin{aligned} &\overline{\eta}_{\mathcal{E}} = \{\overline{\eta}_{e,v=\partial(e)}\}_{e\in\mathcal{E}}, && \eta_{\mathcal{E}} = \{\eta_{e,v=\partial(e)}\}_{e\in\mathcal{E}} \\ &\overline{\eta}_{\mathcal{I}} = \{\overline{\eta}_{i,v}\}_{i\in\mathcal{I},v\in\partial(i)}, && \eta_{\mathcal{I}} = \{\eta_{i,v}\}_{i\in\mathcal{I},v\in\partial(i)} \end{aligned}$$

and correspondingly we set

$$\begin{aligned} \overline{\eta}\cdot\mathcal{L}_{\mathcal{G}}\eta &= \overline{\eta}_{\mathcal{E}}\cdot\mathcal{L}_{\mathcal{E}}\eta_{\mathcal{E}} + \overline{\eta}_{\mathcal{E}}\cdot\mathcal{L}_{\mathcal{E},\mathcal{I}}\eta_{\mathcal{I}} + \overline{\eta}_{\mathcal{I}}\cdot\mathcal{L}_{\mathcal{I},\mathcal{E}}\eta_{\mathcal{E}} + \overline{\eta}_{\mathcal{I}}\cdot\mathcal{L}_{\mathcal{I}}\eta_{\mathcal{I}} \\ &= \left(\overline{\eta}_{\mathcal{I}} + \left(\mathcal{L}_{\mathcal{I}}^{-1}\right)^T(\mathcal{L}_{\mathcal{E},\mathcal{I}})^T\overline{\eta}_{\mathcal{E}}\right)\cdot\mathcal{L}_{\mathcal{I}}\left(\eta_{\mathcal{I}} + \mathcal{L}_{\mathcal{I}}^{-1}\mathcal{L}_{\mathcal{I},\mathcal{E}}\eta_{\mathcal{E}}\right) \\ &\quad + \overline{\eta}_{\mathcal{E}}\cdot\left(\mathcal{L}_{\mathcal{E}} - \mathcal{L}_{\mathcal{E},\mathcal{I}}\mathcal{L}_{\mathcal{I}}^{-1}\mathcal{L}_{\mathcal{I},\mathcal{E}}\right)\eta_{\mathcal{E}}. \end{aligned} \tag{3.8}$$

In analogy to (2.10) the first line just corresponds to the following block matrix representation:

$$\mathcal{L}_{\mathcal{G}} = \begin{pmatrix} \mathcal{L}_{\mathcal{E}} & \mathcal{L}_{\mathcal{E},\mathcal{I}} \\ \mathcal{L}_{\mathcal{I},\mathcal{E}} & \mathcal{L}_{\mathcal{I}} \end{pmatrix} \tag{3.9}$$

up to a different index ordering. Provided the matrix $\mathcal{L}_{\mathcal{I}}$ is invertible, we may define its Schur complement

$$\mathcal{K}_{\mathcal{G}} = \mathcal{L}_{\mathcal{E}} - \mathcal{L}_{\mathcal{E},\mathcal{I}}\mathcal{L}_{\mathcal{I}}^{-1}\mathcal{L}_{\mathcal{I},\mathcal{E}}. \tag{3.10}$$

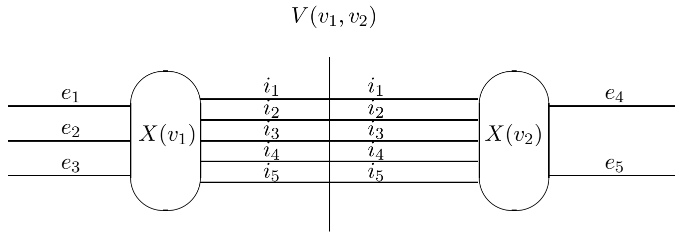

Figure 2. Pictorial description of a two-vertex graph $\mathcal{G}_2$ appearing in the definition of the generalized star product. $X(v_1)$ is an 8×8 matrix, $X(v_2)$ a 7×7 matrix and $V(v_1, v_2)$ a 5×5 matrix. The graph has five external edges $e_1, \ldots, e_5$ and five internal edges $i_1, \ldots, i_5$. The index orderings of $\mathcal{E}\cup\mathcal{I}$ and hence of $\mathcal{I}$ are $\mathcal{E}\cup\mathcal{I} = \{e_1, \ldots, e_5, i_1, \ldots, i_5\}, \mathcal{I} = \{i_1, \ldots, i_5\} = \mathcal{I}(v_1)\cap\mathcal{I}(v_2)$ with $\mathcal{I}(v_1) = \{e_1, e_2, e_3, i_1, \ldots, i_5\}, \mathcal{I}(v_2) = \{i_1, \ldots, i_5, e_4, e_5\}$. $\mathcal{L}_{\mathcal{G}_2}$ is a 15×15 matrix and $\mathcal{L}_{\mathcal{E}}$ a 5×5 matrix.

It will also be convenient to introduce the notation

$$\mathcal{T}_{\mathcal{G}} = \mathcal{L}_{\mathcal{I}} \tag{3.11}$$

in order to indicate the dependence on $\mathcal{G}$.

Theorem 3.2. *If $\mathcal{L}_{\mathcal{I}}$ is invertible, then*

$$\int \mathrm{e}^{-\overline{\eta}\cdot\mathcal{L}_{\mathcal{G}}\eta}\,\mathrm{d}\overline{\eta}_{\mathcal{I}}\,\mathrm{d}\eta_{\mathcal{I}} = \det\mathcal{T}_{\mathcal{G}}\cdot\mathrm{e}^{-\overline{\eta}_{\mathcal{E}}\cdot\mathcal{K}_{\mathcal{G}}\eta_{\mathcal{E}}} \tag{3.12}$$

and thus also $\det\mathcal{L}_{\mathcal{G}} = \det\mathcal{T}_{\mathcal{I}}\cdot\det\mathcal{K}_{\mathcal{G}}$ *hold. In particular, if in addition $\mathcal{L}_{\mathcal{G}}$ is invertible, then $\mathcal{K}_{\mathcal{G}}$ is also invertible and the matrix elements of its inverse are those of the corresponding ones for $\mathcal{L}_{\mathcal{G}}^{-1}$ itself*

$$\left(\mathcal{K}_{\mathcal{G}}^{-1}\right)_{e,v=\partial(e);e',v'=\partial(e')} = \left(\mathcal{L}_{\mathcal{G}}^{-1}\right)_{e,v=\partial(e);e',v'=\partial(e')}. \tag{3.13}$$

Proof. In view of (3.8), this theorem is a direct consequence of lemma 2.1. □

3.1. The generalized star product and two-vertex graphs

In this subsection we will show, how in the case of any two-vertex graph $\mathcal{G}_2$ the above construction of $\mathcal{K}_{\mathcal{G}_2}$ out of $\mathcal{L}_{\mathcal{G}_2}$ is also obtained from the generalized star product introduced in [18, 19]. The vertices are denoted as v_1 and v_2. Figure 2 serves as an illustration. Let the matrix $X(v_1)$, indexed by $\mathcal{I}(v_1)$, be given in a 2×2 block form

$$X(v_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.14}$$

where A is an $n_1\times n_1$ matrix, B an $n_1\times p$ matrix, C an $p\times n_1$ matrix and finally D a $p\times p$ matrix. Here $n_1 = |\mathcal{E}\cap\mathcal{I}(v_1)|$ is the number of external edges e terminating at v_1, that is $\partial(e) = v_1$. $p = n(\mathcal{I})$ the number of internal lines such that $n_1 + p = n(v_1)$. Thus for example A is indexed by $\mathcal{E}\cap\mathcal{I}(v_1)$ while D is indexed by $\mathcal{I}\cap\mathcal{I}(v_1)$. Similarly, write $X(v_2)$, indexed by $\mathcal{I}(v_2) = (\mathcal{I}(v_1))$, in a 2×2 block matrix form

$$X(v_2) = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \tag{3.15}$$

where H is a $p\times p$ matrix, G is a $p\times m_1$ matrix, etc. Here $m_1 = |\mathcal{E}\cap\mathcal{I}(v_2)|$ is the number of external edges e terminating at v_2, that is $\partial(e) = v_2$. Thus for example E is indexed by

$\mathcal{E} \cap \mathcal{I}(v_2)$. Then the $(n_1 + p + m_1 + p) \times (n_1 + p + m_1 + p)$ matrix $\mathcal{L}_{\mathcal{G}_2}$, which is obtained from $X(v_1)$, $X(v_2)$ and $V = V(v_1, v_2)$, takes the form

$$\mathcal{L}_{\mathcal{G}} = \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & -V^{-1} \\ 0 & 0 & E & F \\ 0 & -V & G & H \end{pmatrix} \qquad (3.16)$$

$$\underbrace{\qquad}_{n_1} \quad \underbrace{\qquad}_{p} \quad \underbrace{\qquad}_{m_1} \quad \underbrace{\qquad}_{p}$$

which in a canonical way is indexed by the set $(\mathcal{E} \cup \mathcal{I}) \triangleright \mathcal{V}$. This gives

$$\mathcal{L}_{\mathcal{E}} = \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}, \qquad \mathcal{L}_{\mathcal{E},\mathcal{I}} = \begin{pmatrix} B & 0 \\ 0 & F \end{pmatrix}$$
$$\mathcal{L}_{\mathcal{I},\mathcal{E}} = \begin{pmatrix} C & 0 \\ 0 & G \end{pmatrix}, \qquad \mathcal{T}_{\mathcal{G}_2} = \mathcal{L}_{\mathcal{I}} = \begin{pmatrix} D & -V^{-1} \\ -V & H \end{pmatrix} \qquad (3.17)$$

where $\mathcal{L}_{\mathcal{E}}$ is a $|\mathcal{E}| \times |\mathcal{E}|$ matrix, etc. Therefore by (3.10)

$$\mathcal{K}_{\mathcal{G}_2} = \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} B & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} D & -V^{-1} \\ -V & H \end{pmatrix}^{-1} \begin{pmatrix} C & 0 \\ 0 & G \end{pmatrix}$$
$$= \begin{pmatrix} A - B(D - V^{-1}H^{-1}V)^{-1}C & -B(D - V^{-1}H^{-1}V)^{-1}V^{-1}H^{-1}G \\ -F(H - VD^{-1}V^{-1})^{-1}VD^{-1}C & E - F(H - VD^{-1}V^{-1})^{-1}G \end{pmatrix}. \qquad (3.18)$$

We have used (2.16) by which

$$\begin{pmatrix} D & -V^{-1} \\ -V & H \end{pmatrix}^{-1} = \begin{pmatrix} (D - V^{-1}H^{-1}V)^{-1} & (D - V^{-1}H^{-1}V)^{-1}V^{-1}H^{-1} \\ (H - VD^{-1}V^{-1})^{-1}VD^{-1} & (H - VD^{-1}V^{-1})^{-1} \end{pmatrix}$$

and we have assumed the matrix $\mathcal{T}_{\mathcal{G}_2} = \mathcal{L}_{\mathcal{I}}$ to be invertible. Alternatively, by theorem 3.2, if $\mathcal{L}_{\mathcal{G}_2}^{-1}$ is written in 4×4 block form like $\mathcal{L}_{\mathcal{G}_2}$

$$\mathcal{L}_{\mathcal{G}_2}^{-1} = \begin{pmatrix} \alpha & \cdot & \beta & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \gamma & \cdot & \delta & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

then

$$\mathcal{K}_{\mathcal{G}_2}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad (3.19)$$

holds. This can be checked by calculating the inverse of $\mathcal{L}_{\mathcal{G}_2}$, a tedious but straightforward calculation using (2.16) iteratively. Moreover we have

Lemma 3.3. *Assume $\mathcal{T}_{\mathcal{G}_2} = \mathcal{L}_{\mathcal{I}}$ to be invertible, such that $\mathcal{K}_{\mathcal{G}_2}$ is well defined. Then the generalized star product $X(v) \star_{V(v,v')} X(v')$ as introduced in [18] is also well defined and both these quantities are equal.*

Proof. In the present notation, where we recall $V = V(v, v')$,

$$X(v) \star_{V(v,v')} X(v') = \begin{pmatrix} A + BK_2HVC & BK_2G \\ FK_1C & E + FK_1DV^{-1}G \end{pmatrix} \qquad (3.20)$$

holds with

$$K_1 = (\mathbb{I} - VDV^{-1}H)^{-1}V \quad = V(\mathbb{I} - DV^{-1}HV)^{-1}$$
$$K_2 = (\mathbb{I} - V^{-1}HVD)^{-1}V^{-1} = V^{-1}(\mathbb{I} - HVDV^{-1})^{-1}, \qquad (3.21)$$

see section 4 in [18] and section 3 in [19]. We use the relations

$$(D - V^{-1}H^{-1}V)^{-1} = -(\mathbb{I} - V^{-1}HVD)^{-1}V^{-1}HV = -K_2HV$$
$$(H - VD^{-1}V^{-1})^{-1} = -(\mathbb{I} - VDV^{-1}H)^{-1}VDV^{-1} = -K_1DV^{-1},$$

insert this into (3.18). Comparison with (3.20) gives the claim. □

3.2. The generalized star product and arbitrary graphs

We are now able to extend this comparison to the case where the graph has more than two vertices. The idea for this alternative is to carry out the integrations in (3.12) in steps while using iteratively the Grassmannian–Gaussian construction of the star product as given in the previous subsection. This proof will give a more explicit representation of $\mathcal{K}_{\mathcal{G}}$ and $\mathcal{T}_{G}$ in terms of the original data $\mathcal{G}$, $\underline{X}_{\mathcal{G}}$ and $\underline{V}_{\mathcal{G}}$. It is important to recall that the data uniquely fix $\mathcal{L}_{\mathcal{G}}$. So this alternative proof will be by induction on the number of vertices, by which we will construct a sequence of connected graphs $\mathcal{G}_l$ with l vertices such that $\mathcal{G}_{l=n(\mathcal{V})} = \mathcal{G}$. Similarly we will provide inductively $\mathcal{K}_l$—with $\mathcal{K}_{\mathcal{G}_{l=n(\mathcal{V})}} = \mathcal{K}_{\mathcal{G}}$—and $\mathcal{T}_l$, the last one will be given recursively in the form

$$\mathcal{T}_l = \mathcal{T}_{l-1} \oplus \mathcal{T}^l \tag{3.22}$$

with suitable $\mathcal{T}^l$ and where by definition $\mathcal{T}^1 = 1$. Here and in what follows we make the notational convention that for any two square matrices M_1 and M_2 their direct sum $M_1 \oplus M_2$ is identified with the 2×2 block matrix

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

We first construct the $\mathcal{G}_l$ inductively. As for the case $l = 1$, choose any vertex and call it v_1. Let $\mathcal{G}_1$ denote a star graph with $n(v_1)$ external lines labeled by $\mathcal{I}(v_1)$ as for the graph $\mathcal{G}$ itself. Assume that we have constructed the connected graph $\mathcal{G}_l$ with the set of vertices $\mathcal{V}_l = \{v_1, \ldots, v_l\}$, named like those of $\mathcal{G}$. Also the edges $i \in \mathcal{G}_l$ with $v_k \in \partial(i)$ are in one-to-one correspondence with the edges in $\mathcal{G}$ having v_k in their boundary. Thus we may use $\mathcal{I}(v_k)$ to index these edges. Furthermore the sets of external and internal edges in $\mathcal{G}_l$ are such that

$$\begin{aligned} &\mathcal{E}_l \cup \mathcal{I}_l = \cup_{1\leqslant k\leqslant l}\mathcal{I}(v_k) \\ &\mathcal{I}_l = \cup_{1\leqslant k\neq k'\leqslant l}(\mathcal{I}(v_k) \cap \mathcal{I}(v_{k'})). \end{aligned} \tag{3.23}$$

To obtain $\mathcal{G}_{l+1}$ from $\mathcal{G}_l$, observe there is a vertex in $\mathcal{G}$, denoted v_{l+1}, such that

$$\bar{n}(v_{l+1}) = \sum_{k=1}^{l} n(v_k, v_{l+1}) > 0.$$

$\mathcal{G}_{l+1}$ is obtained as follows. Take $\mathcal{G}_l$ and a single-vertex graph with vertex denoted by v_{l+1} and with $\bar{n}(v_{l+1})$ edges emanating. Call this graph $\bar{\mathcal{G}}(v_{l+1})$. Label its edges by the elements in $\cup_{k\leqslant l}(\mathcal{I}(v_k) \cap \mathcal{I}(v_l))$. Glue each edge $i \in \mathcal{I}(v_k) \cap \mathcal{I}(v_l)$ in $\bar{\mathcal{G}}(v_{l+1})$ to the edge with the same index in $\mathcal{G}_l$. In case $\mathcal{G}$ is a metric graph with set of internal edge lengths $\underline{a}$, give the resulting internal edge i in $\mathcal{G}_{l+1}$ the length a_i. To sum up, the resulting graph $\mathcal{G}_{l+1}$ has $n(v_k, v_{l+1})$ edges connecting v_k with v_{l+1} in $\mathcal{G}_{l+1}$. In total $\mathcal{G}_{l+1}$ thus obtained has edges which also are of the form (3.23) with l being replaced by $l + 1$. This concludes the inductive construction of the graphs and gives $\mathcal{G}$ as $\mathcal{G}_{l=n(\mathcal{V})}$.

$\mathcal{V}_l$ can be viewed as a subset of $\mathcal{V}$ and that by (3.23) $\mathcal{E}_l \cup \mathcal{I}_l$ can be viewed as a subset of both $\mathcal{E}_{l+1} \cup \mathcal{I}_{l+1}$ and $\mathcal{E} \cup \mathcal{I}$. Similarly, $\mathcal{I}_l$ can be viewed as a subset of both $\mathcal{I}_{l+1}$ and $\mathcal{I}$. For the

$\mathcal{E}_l$ similar relations, however, are not valid, since an edge in $\mathcal{E}_{l-1}$ can turn into an edge in $\mathcal{I}_l$. More explicitly we introduce the sets

$$\bar{\mathcal{I}}_l = \mathcal{E}_{l-1} \cap \mathcal{I}_l = \cup_{k:k<l} \left(\mathcal{I}(v_k) \cap \mathcal{I}(v_l)\right) \tag{3.24}$$

which will be used in the following. They satisfy

$$\bar{\mathcal{I}}_l \cap \bar{\mathcal{I}}_{l'} = \emptyset, \quad l \neq l'; \qquad \cup_{1 \leqslant l \leqslant n(\mathcal{V})} \bar{\mathcal{I}}_l = \mathcal{I}. \tag{3.25}$$

Pictorially this construction can be understood as follows. $\mathcal{G}_l$ is obtained from $\mathcal{G}$ by cutting any internal edge, which connects any vertex $v_k (1 \leqslant k \leqslant l)$ to any vertex v different from $v_{k'} (1 \leqslant k' \leqslant l)$. Any such edge is then replaced by an infinite half-line. As a consequence of this discussion

$$\bar{\mathcal{I}}_l \triangleright \mathcal{V}_l \subseteq (\mathcal{E}_l \cup \mathcal{I}_l) \triangleright \mathcal{V}_l \subset (\mathcal{E}_{l+1} \cup \mathcal{I}_{l+1}) \triangleright \mathcal{V}_{l+1} \subseteq (\mathcal{E} \cup \mathcal{I}) \triangleright \mathcal{V}, \qquad 1 \leqslant l \leqslant n(\mathcal{V}) - 1, \tag{3.26}$$

which induces an ordering on these sets. In order to construct the $\mathcal{K}_l$ we introduce the matrices

$$\overline{V}(v_l) = V(v_1, v_l) \oplus V(v_2, v_l) \ldots \oplus V(v_{l-1}, v_l), \qquad l \geqslant 2 \tag{3.27}$$

which are invertible. Thus in the example given by figure 1 $\overline{V}(v_{l=3})$ is a 3×3 matrix.

Set $\mathcal{K}_1 = X(v_1)$ and inductively

$$\mathcal{K}_{l+1} = \mathcal{K}_l \star_{\overline{V}(v_{l+1})} X(v_{l+1}). \tag{3.28}$$

In particular, $\mathcal{K}_2$ is just as given by lemma 3.3. We note that the invertibility of a certain matrix is necessary, see the discussion of $\mathcal{T}_{\mathcal{G}_2}$ in section 3.1. So if the invertibility of certain matrices holds—see also below—the associativity of the generalized star product [18] gives

$$\mathcal{K}_l = X(v_1) \star_{V(v_1, v_2)} X(v_2) \star_{\overline{V}(v_3)} X(v_3) \ldots \star_{\overline{V}(v_{n(l)})} X(v_l). \tag{3.29}$$

We now repeat this construction by performing Grassmann integration over

$$\mathrm{e}^{-\overline{\eta} \cdot \mathcal{L}_{\mathcal{G}} \eta}$$

in steps. In order to do this we view $\mathcal{G}_l$ as a single-vertex graph with a vertex denoted by $\bar{v}_l$ and with edges labeled by $\mathcal{E}_l$. Combine it with the single-vertex graph $\bar{\mathcal{G}}(v_{l+1})$ and with the connecting matrix given as $V(\bar{v}_l, v_{l+1}) = \overline{V}(v_{l+1})$. Correspondingly we take as data for $\mathcal{G}_{l+1}$

$$\underline{X}_{l+1} = \{\mathcal{K}_l, X(v_{l+1})\}, \qquad \underline{V}_{l+1} = \{V(\bar{v}_l, v_{l+1}) = \overline{V}(v_{l+1})\} \tag{3.30}$$

and out of this we form the fermionic Lagrangean

$$\overline{\eta} \cdot \mathcal{L}_{l+1} \eta = \overline{\eta} \cdot \mathcal{K}_{\mathcal{G}_l} \eta + \overline{\eta} \cdot X(v_{l+1}) \eta - \overline{\eta} \cdot \overline{V}(v_{l+1}) \eta. \tag{3.31}$$

In order not to burden the notation, we have not stated explicitly, which Grassmann variables out of the set (3.3) are involved. Indeed, those which are involved can be read off the index set associated with the matrices $\mathcal{K}_l$, $X(v_{l+1})$ and $\overline{V}(v_{l+1})$.

With this notational convention and by lemma 3.3 we obtain

$$\det \mathcal{T}^{l+1} \cdot \mathrm{e}^{-\overline{\eta} \cdot \mathcal{K}_{l+1} \eta} = \int \mathrm{e}^{-\overline{\eta} \cdot \mathcal{L}_{l+1} \eta} \, \mathrm{d}\overline{\eta}_{\bar{\mathcal{I}}_{l+1}} \, \mathrm{d}\eta_{\bar{\mathcal{I}}_{l+1}}. \tag{3.32}$$

where $\mathcal{T}^{l+1} = (\mathcal{L}_{l+1})_{\bar{\mathcal{I}}_{l+1}}$—in an adaption of the notation used in (3.8)—is invertible. We recall that by definition $\mathcal{T}^1 = 1$. This concludes the recursive construction of $\mathcal{G}_l$, $\mathcal{K}_l$ and

$$\mathcal{T}_l = \oplus_{k=1}^{l} \mathcal{T}^k. \tag{3.33}$$

We iterate recursion (3.32) in combination with recursion (3.31), use (3.22), (3.25) and (3.29) to obtain

$$\det \mathcal{T}_{l=n(\mathcal{V})} \cdot \mathrm{e}^{-\overline{\eta}\cdot\mathcal{K}_{l=n(\mathcal{V})}\eta} = \int \mathrm{e}^{-\overline{\eta}\cdot\mathcal{L}_{\mathcal{G}}\eta}\, \mathrm{d}\overline{\eta}_{\mathcal{I}}\, \mathrm{d}\eta_{\mathcal{I}}. \tag{3.34}$$

Comparison with (3.12), while keeping remark 1 in mind, gives the main result of this paper.

Theorem 3.4. *Assume that the matrices $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{T}^{l}$ are all invertible, such that all $\mathcal{T}_l$ are also invertible. Then the quantities $\mathcal{K}_{\mathcal{G}}$ and* $\det \mathcal{T}_{\mathcal{G}}$ *as given by (3.12) are equal to $\mathcal{K}_{l=n(\mathcal{V})}$ and* $\det \mathcal{T}_{l=n(\mathcal{V})}$, *respectively, where $\mathcal{K}_{l=n(\mathcal{V})}$ and $\mathcal{T}_{l=n(\mathcal{V})}$ are defined by (3.29) and (3.33).*

Without proof we state that under the assumptions stated actually $\mathcal{T}_{\mathcal{G}} \simeq \mathcal{T}_{l=n(\mathcal{V})}$ holds.

Corollary 3.5. *For a given graph $\mathcal{G}$ let the data $\underline{X}_{\mathcal{G}}$ and $\underline{V}_{\mathcal{G}}$ consist of unitaries. Under the corresponding invertibility assumptions, $\mathcal{K}_{\mathcal{G}}$ is also unitary as are in fact all $\mathcal{K}_{\mathcal{G}_l}$.*

Proof. This follows from the theorem and the results in [18, 19]. □

4. Conclusions

The result presented in this article is relevant in the context of quantum scattering theory on metric graphs. So the main purpose of this article is mainly pedagogical. The aim was to show how the scattering matrix can be obtained from basic building blocks by simple and well known mathematical techniques. Indeed, in this context $\mathcal{K}_{\mathcal{G}}$ is the scattering matrix at a fixed energy E associated with the entire metric graph $\mathcal{G}$. The metric enters through the connecting matrices given in the form (3.2). The other building blocks, that is the vertex matrices, are the single vertex scattering matrices at the same energy E, see Remark 3.1. These scattering matrices have been used to study spin transport and conductance on quantum graphs, see [35–37]. More recently, another way of obtaining the scattering matrix by an iterative procedure, again equivalent to the generalized star product procedure, has been given in [37]. It involves the use of RT-algebras. Relation (3.36) in [20] provides yet another way to obtain this scattering matrix in terms of the single vertex scattering matrices and connecting matrices. In addition in [20] a series expansion of every matrix element is given. This expansion is indexed by so called walks $\mathbf{w}$ with length $|\mathbf{w}|$ and has expansion coefficients of the form $\exp(\mathrm{i}\sqrt{E}|\mathbf{w}|)$ times a monomial in the single vertex scattering matrix elements. It has been used to formulate a new approach to the traveling salesman problem.

Acknowledgments

The authors would like to thank M Karowski for valuable discussions. RS and AS are supported in part by the Humboldt Foundation.

Appendix A. The generalized star product and Gaussian integrals

In this appendix we give a *bosonic* discussion using Gaussian distributions. We start by recalling some basic facts about Gaussian distributions, also in order to establish notation and for the sake of comparison with our *fermionic* discussion. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ denote elements in $\mathbb{R}^n$ and set

$$y \cdot x = \sum_{i=1}^{n} y_i x_i = x \cdot y$$

and for integration $\mathrm{d}x = \prod_{i=1}^{n} \mathrm{d}x_i$ denotes the infinitesimal volume element on $\mathbb{R}^n$. Let A be a real symmetric matrix, which in addition is *positive (definite)*

$$x \cdot Ax = \sum_{i,j=1}^{n} x_i X_{ij} x_j > 0, \qquad x \neq 0$$

and then we write $A > 0$. Then also $\det A > 0$ and in addition A^{-1} exists, is real, symmetric and positive. $\mathbb{I}$ denotes the unit matrix in the given context and $A > A'$ means $x \cdot Ax > x \cdot A'x$ for all $x \neq 0$. If $\kappa\mathbb{I} < A < \mu\mathbb{I}$, then $\mu^{-1}\mathbb{I} < A^{-1} < \kappa^{-1}\mathbb{I}$.

We make the following notational convention. Here and in what follows, whenever $x \in \mathbb{R}^n$ stands to the right of an $m \times n$ matrix A, then it is viewed as a column vector. The outcome Ax will also be interpreted as a column vector in $\mathbb{R}^m$. When x stands to the left of A it will be viewed as a row vector.

Define the Gauss distribution with covariance $A > 0$ via its probability measure

$$\mathrm{d}\mu_A(x) = \frac{(\det A)^{1/2}}{(2\pi)^{n/2}} \mathrm{e}^{-\frac{1}{2} x \cdot Ax} \, \mathrm{d}x \tag{A.1}$$

on $\mathbb{R}^n$. That this is a probability measure follows from

$$\int_{x \in \mathbb{R}^n} \mathrm{e}^{-\frac{1}{2} x \cdot Ax} \, \mathrm{d}x = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}} \tag{A.2}$$

and is the analogue of (2.7).

The first two moments of the measure $\mathrm{d}\mu_A(x)$ are

$$\int_{\mathbb{R}^n} x_i \, \mathrm{d}\mu_A(x) = 0 \qquad \int_{\mathbb{R}^n} x_i x_j \, \mathrm{d}\mu_A(x) = A^{-1}_{ij} \tag{A.3}$$

which are the analogues of (2.8).

Write any $x \in \mathbb{R}^n$ as $x = (x^{(1)}, x^{(2)})$ with $x^{(1)} = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $x^{(2)} = (x_{p+1}, \ldots, x_n) \in \mathbb{R}^{n-p}$ and set

$$x^{(1)} \cdot A_{11} x^{(1)} = \sum_{i,j \leqslant p} x_i A_{ij} x_j, \qquad x^{(1)} \cdot A_{11} x^{(2)} = \sum_{i \leqslant p, p+1 \leqslant j} x_i A_{ij} x_j$$

$$x^{(2)} \cdot A_{21} x^{(1)} = \sum_{p+1 \leqslant i, j \leqslant p} x_i A_{ij} x_j, \qquad x^{(2)} \cdot A_{22} x^{(2)} = \sum_{p+1 \leqslant i, j \leqslant p} x_i A_{ij} x_j$$

such that we have the decomposition

$$x \cdot Ax = x^{(1)} \cdot A_{11} x^{(1)} + x^{(1)} \cdot A_{12} x^{(2)} + x^{(2)} \cdot A_{21} x^{(1)} + x^{(2)} \cdot A_{22} x^{(2)}.$$

In other words, we use the 2×2 block decomposition (2.10) of the matrix A. Since A is assumed to be positive definite, so are A_{11} and A_{22} and their inverses. Also A_{21} is the transpose of A_{12}. So $\widehat{A} = A_{11} - A_{12} A_{22}^{-1} A_{21}$ is a well-defined and symmetric $p \times p$ matrix. In fact it is positive definite, see, e.g. [34]. Actually we need a stronger result.

Lemma A.1. *If $A > \kappa\mathbb{I}$ with $\kappa > 0$ holds, then also $\widehat{A} > \kappa\mathbb{I}$ is valid for the Schur complement of A_{22}.*

Proof. Under the assumption $0 < A^{-1} < \kappa^{-1}\mathbb{I}$, hence also $0 < \widehat{A}^{-1} < \kappa^{-1}\mathbb{I}$ due to (2.16). Taking the inverse gives $\widehat{A} > \kappa\mathbb{I}$. □

We leave the proof of the following lemma to the reader. It is the analogue of lemma 2.1.

Lemma A.2. *The following relation holds*

$$\int_{x^{(2)}\in\mathbb{R}^{n-p}} \mathrm{e}^{-\frac{1}{2}x\cdot Ax}\,\mathrm{d}x^{(2)} = \frac{(2\pi)^{n-p/2}}{(\det A_{22})^{1/2}}\,\mathrm{e}^{-\frac{1}{2}x^{(1)}\cdot\widehat{A}x^{(1)}}. \tag{A.4}$$

Since $\widehat{A}$ is positive definite, we may integrate (A.4) over $x^{(1)}$ and $\det A = \det A_{22}\cdot\det\widehat{A}$ follows, which is relation (2.14), however, in the restricted context of positive A.

We turn to the generalized star product and a way to obtain it through Gaussian integrals. Consider $X(v_1)$ and $X(v_2)$ of the form (3.14) and (3.15). We use the notation employed in this context.

Proposition A.3. *Assume $X(v_1) > \kappa\mathbb{I}, X(v_2) > \kappa\mathbb{I}$ with $\kappa > 1$ and let $V(v_1, v_2)$ be an orthogonal $p\times p$ matrix. Then $X(v_1)\star_{V(v_1,v_2)} X(v_2) > (\kappa-1)\mathbb{I}$.*

Proof. Let $\mathcal{L}_{\mathcal{G}_2}$ be as in (3.16) but now indexed from 1 to n_1+p+m_1+p. Also let the orthogonal matrix $V = V(v_1,v_2) = V(v_2,v_1)^{-1} = V(v_2,v_1)^T$ be indexed from 1 to p. For $0\neq x\in\mathbb{R}^{n_1+m_1+2p}$ by Schwarz inequality

$$\begin{aligned} x\cdot\mathcal{L}_{\mathcal{G}_2}x = &\sum_{1\leqslant i,j\leqslant n_1+p} x_i X(v_1)_{ij}x_j + \sum_{n_1+p+1\leqslant i,j\leqslant n_1+m_1+2p} x_i X(v_1)_{ij}x_j \\ &- \sum_{1\leqslant i,j\leqslant p} x_{n_1+i}V_{ij}x_{n_1+p+m_1+j} - \sum_{1\leqslant i,j\leqslant p} x_{n_1+p+m_1+i}V_{ji}x_{n_1+j} > (\kappa-1)x\cdot x. \end{aligned}$$

But $X(v_1)\star_{V(v_1,v_2)} X(v_2)$ is a Schur complement by the discussion in subsection 3.1, that is $X(v_1)\star_{V(v_1,v_2)} X(v_2) = \mathcal{K}_{\mathcal{G}_2}$, and so the claim follows from lemma A.1. □

Write $x = (z^{(1)}, z^{(2)})\in\mathbb{R}^{n_1+m_1+2p}$ with $z^{(1)} = (x_1,\ldots,x_{n_1},x_{n_1+2p+1},\ldots,x_{n_1+2p+m_1})\in\mathbb{R}^{n_1+m_1}$ and $z^{(2)} = (x_{n_1+1},\ldots,x_{n_1+2p})\in\mathbb{R}^{2p}$. Then with the notation used in section 3.1 and in the proof of the lemma we obtain

Lemma A.4. *With the assumptions as in lemma A.2 the relation*

$$\int_{z^{(2)}\in\mathbb{R}^{2p}} \mathrm{e}^{-\frac{1}{2}x\cdot\mathcal{L}_{\mathcal{G}_2}x}\,\mathrm{d}z^{(2)} = \frac{(2\pi)^p}{(\det\mathcal{T}_{\mathcal{G}_2})^{1/2}}\,\mathrm{e}^{-\frac{1}{2}z^{(1)}\cdot\mathcal{K}_{\mathcal{G}_2}z^{(1)}} \tag{A.5}$$

holds.

We turn to an arbitrary graph $\mathcal{G}$ with data $\underline{X}_{\mathcal{G}}$ and $\underline{V}_{\mathcal{G}}$ with the property that each $X(v)$ is positive definite and each $V(v,v')$ is orthogonal. Introduce the variable

$$x = \{x_{i,v}\}_{i,v\in(\mathcal{E}\cup\mathcal{I})\triangleright\mathcal{V}} = \{x_{i,v}\}_{i,v:i\in\mathcal{I}(v),v\in\mathcal{V}}\in\mathbb{R}^{|\mathcal{E}|+2|\mathcal{I}|},$$

let the matrices $\mathcal{L}(v)$ and $\mathcal{L}(v,v')$ be as in (3.6) and set

$$\begin{aligned} x\cdot\mathcal{L}(v)x &= \sum_{i,j\in\mathcal{I}(v)} x_{i,v}X(v)_{ij}x_{j,v} \\ x\cdot\mathcal{L}(v,v')x &= -\sum_{i,j\in\mathcal{I}(v)\cap\mathcal{I}(v')} x_{i,v}V(v,v')_{ij}x_{j,v'}. \end{aligned} \tag{A.6}$$

Define the *quadratic, bosonic Lagrangian* as

$$x\cdot\mathcal{L}_{\mathcal{G}}x = \sum_v x\cdot\mathcal{L}(v)x - \sum_{v\neq v'} x\cdot\mathcal{L}(v,v')x. \tag{A.7}$$

We decompose x into its *exterior* and *interior components*, that is $x = (x_{\mathcal{E}}, x_{\mathcal{I}})$ with

$$x_{\mathcal{E}} = \{x_{e,v=\partial(e)}\}_{e\in\mathcal{E}},\qquad x_{\mathcal{I}} = \{x_{i,v}\}_{i\in\mathcal{I}(v),v\in\mathcal{V}}.$$

and correspondingly we get

$$\begin{aligned} x \cdot \mathcal{L}_{\mathcal{G}} x &= x_{\mathcal{E}} \cdot \mathcal{L}_{\mathcal{E}} x_{\mathcal{E}} + x_{\mathcal{E}} \cdot \mathcal{L}_{\mathcal{E},\mathcal{I}} x_{\mathcal{I}} + x_{\mathcal{I}} \cdot \mathcal{L}_{\mathcal{I},\mathcal{E}} x_{\mathcal{E}} + x_{\mathcal{I}} \cdot \mathcal{L}_{\mathcal{I}} x_{\mathcal{I}} \\ &= (x_{\mathcal{I}} + (\mathcal{L}_{\mathcal{I}})^{-1T} (\mathcal{L}_{\mathcal{E},\mathcal{I}})^T x_{\mathcal{E}}) \cdot \mathcal{L}_{\mathcal{I}} (x_{\mathcal{I}} + (\mathcal{L}_{\mathcal{I}})^{-1} \mathcal{L}_{\mathcal{I},\mathcal{E}} x_{\mathcal{E}}) \\ &\quad + x_{\mathcal{E}} \cdot (\mathcal{L}_{\mathcal{E}} - \mathcal{L}_{\mathcal{E},\mathcal{I}} (\mathcal{L}_{\mathcal{I}})^{-1} \mathcal{L}_{\mathcal{I},\mathcal{E}}) x_{\mathcal{E}}. \end{aligned} \tag{A.8}$$

The following extension of lemma A.4 to arbitrary graphs is valid.

Proposition A.5. *Given data* $\underline{X}_{\mathcal{G}} = \{X(v)\}_{v \in \mathcal{G}}$ *and* $\underline{V}_{\mathcal{G}} = \{V(v, v')\}_{v \neq v' \in \mathcal{V}}$ *with* $X(v) > (n(\mathcal{V}) - 1)\mathbb{I}$ *and orthogonal* $V(v, v')$*, the* $|\mathcal{E}| \times |\mathcal{E}|$ *matrix* $\mathcal{K}_{\mathcal{G}}$ *defined by*

$$\int_{x_{\mathcal{I}} \in \mathbb{R}^{2|\mathcal{I}|}} \mathrm{e}^{-\frac{1}{2} x \cdot \mathcal{L}_{\mathcal{G}} x} \, \mathrm{d}x_{\mathcal{I}} = \frac{(2\pi)^{|\mathcal{I}|}}{(\det \mathcal{T}_{\mathcal{G}})^{1/2}} \, \mathrm{e}^{-\frac{1}{2} x_{\mathcal{E}} \cdot \mathcal{K}_{\mathcal{G}} x_{\mathcal{E}}} \tag{A.9}$$

is positive definite.

Proof. We use representation (3.29) and lemma A.1 repeatedly. Observe that $\overline{V}(v_{l+1})$ defined by (3.27) is also orthogonal. □

References

[1] Aktosun T 1992 A factorization of the scattering matrix for the Schrödinger equation and for the wave equation in one dimension *J. Math. Phys.* **33** 3865–9

[2] Beennakker C W J 1997 Random theory of quantum transport *Rev. Mod. Phys.* **69** 731–808

[3] Berezin F A 1966 *The Method of Second Quantization* (New York: Academic)

[4] Berkolaiko G, Carlson R, Fulling S A and Kuchment P (ed) 2006 *Quantum Graphs and Their Applications* (*Contemporary Mathematics* vol 415) (Providence, RI: American Mathematical Society)

[5] Chalker J and Coddington P 1988 Percolation, quantum tunneling and the integer Hall effect *J. Phys. C: Solid State Phys.* **21** 2665–79

[6] Datta S 1995 *Electronic Transport in Mesoscopic Systems* (Cambridge: Cambridge University Press)

[7] Dorokhov O N 1982 Transmission coefficient and the localization length of an electron in N bound disordered chains *JETP Lett.* **36** 318–21

[8] Dorokhov O N 1983 Electron localization in a multichannel conductor *Sov. Phys.—JETP* **58** 606–15

[9] Dorokhov O N 1984 On the coexistence of localized and extended electronic states in the metallic phase *Solid State Commun.* **51** 381–4

[10] Dorokhov O N 1988 Solvable model of multichannel localization *Phys. Rev.* B **37** 10526–41

[11] Exner P, Keating J, Kuchment P, Sunada T and Teplyaev A (ed) 2008 *Analysis on Graphs and its Applications* (*Proc. Symposia in Pure Mathematics* vol 77) (Providence, RI: American Mathematical Society)

[12] Exner P and Tater M 1998 Evanescent modes in a multiple scattering factorization *Czech. J. Phys.* **48** 617–24

[13] Gill P E, Murray W and Wright M H 1991 *Numerical Linear Algebra and Optimization* (Redwood City, CA: Addison-Wesley)

[14] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill)

[15] Kostrykin V and Schrader R 1994 Cluster properties of one particle Schrödinger operators *Rev. Math. Phys.* **6** 833–53

[16] Kostrykin V and Schrader R 1998 Cluster properties of one particle Schrödinger operators: II *Rev. Math. Phys.* **10** 627–83

[17] Kostrykin V and Schrader R 1999 Scattering theory approach to random Schrödinger operators in one dimension *Rev. Math. Phys.* **11** 187–242

[18] Kostrykin V and Schrader R 1999 Kirchhoffs rule for quantum wires *J. Phys. A: Math. Gen.* **32** 595–630

[19] Kostrykin V and Schrader R 2001 The generalized star product and the factorization of scattering matrices on graphs *J. Math. Phys.* **42** 1563–98

[20] Kostrykin V and Schrader R 2006 The inverse scattering problem for metric graphs and the traveling salesman problem arXiv:math-ph/0603010

[21] Kowal D, Sivan U, Entin-Wohlman O and Imry Y 1990 Transmission through multiply-connected wire systems *Phys. Rev.* B **42** 9009–18

[22] Kuchment P 2004 Quantum graphs: I. Some basic structures *Waves Random Media* **14** S107–28

[23] Mello P A, Pereyra P and Kumar N 1988 Macroscopic approach to multichannel disordered conductors *Ann. Phys.* **181** 290–317

[24] Rozman M G, Reineker P and Tehver R 1994 One-dimensional scattering: recurrence relations and differential equations *Phys. Rev.* A **49** 3310–21
[25] Redheffer R 1961 Difference equations and functional equations in transmission-line theory *Modern Mathematics for the Engineer* ed E F Beckenbach (New York: McGraw-Hill)
[26] Redheffer R 1962 On the relation of transmission line theory to scattering and transfer *J. Math. Phys.* **41** 1–41
[27] de Bianchi M S and Ventra M D 1995 On the number of states bound by one-dimensional finite periodic potentials *J. Math. Phys.* **36** 1753–64
[28] de Bianchi M S and Ventra M D 1995 Differential equations and factorization property for the one-dimensional Schrödinger equation with position-dependent mass *Eur. J. Phys.* **16** 260–5
[29] Sedrakyan A 2003 Action formulation of the network model of plateau–plateau transitions in the quantum Hall effect *Phys. Rev.* B **68** 235329
[30] Sedrakyan A 2008 On the continuum limit of the Chalker–Coddington network model *Invited talk at the 411. Heraeus Seminar and School Network Models in Quantum Physics, Jacobs Universität Bremen, 20–25 July 2008*
[31] Stone A D, Mello P A, Muttalib K and Pichard J-L 1991 Random matrix theory and maximum entropy models for disordered conductors *Mesoscopic Phenomena in Solids* ed B L Altschuler, P A Lee and R A Webb (Amsterdam: North-Holland) pp 369–448
[32] Tong B Y 1968 Electronic structure of one-dimensional binary alloys *Phys. Rev.* **175** 710–722
[33] Weinberg S 1995 *The Quantum Theory of Fields I* (Cambridge: Cambridge University Press)
[34] Zhang F 2005 *The Schur Complement and Its Applications* (Berlin: Springer)
[35] Bellazzini B, Mintchev M and Sorba P 2007 Bosonization and scale invariance on quantum wires *J. Phys. A: Math. Theor.* **40** 2485–507 (arXiv:hep-th/0611090)
[36] Bellazzini B, Burrello M, Mintchev M and Sorba P 2008 Quantum field theory on star graphs *Proc. Symp. Pure Math.* **77** 639–56
[37] Ragoucy E 2009 Quantum field theory on quantum graphs and application to their conductance LAPTH-1304/09 (arXiv:0901.2431)